ALGORITHMS FOR OUTPUT FEEDBACK, MULTIPLE-MODEL, AND DECENTRALIZED CONTROL PROBLEMS

Nesim Halyo and John R. Broussard
Information and Control Systems, Incorporated
Hampton, Virginia

First Annual NASA Aircraft Controls Workshop
NASA Langley Research Center
Hampton, Virginia
October 25-27, 1983

ABSTRACT

The optimal stochastic output feedback, multiple-model, and decentralized control problems with dynamic compensation are formulated and discussed. Algorithms for each problem are presented, and their relationship to a basic output feedback algorithm is discussed. An aircraft control design problem is posed as a combined decentralized, multiple-model, output feedback problem. A control design is obtained using the combined algorithm. An analysis of the design is presented.

ADVANTAGES OF STOCHASTIC OUTPUT FEEDBACK

The stochastic optimal output feedback problem [1-8] is a significant extension of the "full-state feedback" LQG problem [9]. Its formulation addresses some important limitations encountered in practical systems and provides a flexibility useful in configuring the control law for ease of implementation. Some of the advantages of the stochastic output feedback problem are shown below. Output feedback introduces a rich class of control law structures which can be used in modern control designs.

- DESIGNER CAN SELECT THE STATES FOR FEEDBACK
- PROVIDES A METHOD TO DESIGN OUTER LOOP CONTROL LAWS
- ACCOUNTS FOR ACTUATOR DYNAMICS WITHOUT NECESSITY FOR ACTUATOR STATE FEEDBACK
- ACCOUNTS FOR PHASE SHIFTS INTRODUCED BY PREFILTERS AND OTHER ESTIMATORS WITHOUT NECESSITY OF FEEDBACK
- PROVIDES A SYSTEMATIC METHOD TO INCREASE OR DECREASE GAINS BY ADJUSTING PLANT AND MEASUREMENT NOISE COVARIANCES
- PROVIDES CONSIDERABLE FLEXIBILITY IN THE CONTROL STRUCTURE IN A MODERN CONTROL SETTING

FORMULATION OF THE STOCHASTIC OUTPUT FEEDBACK PROBLEM

The discrete stochastic optimal output feedback problem is formulated below. The control U_k feeds back the output Y_k through a constant gain matrix K. The term $\mathcal{D}$ is the set of gains K for which $J_N(K)$ converges to a finite value $J(K)$. The term $\mathcal{S}$ is the set of gains which stabilizes the closed-loop system. The optimization problem can be posed as: Find a stabilizing gain K^* ($K^* \in \mathcal{S}$) which minimizes the cost $J(K)$, i.e., $J(K^*) \leq J(K)$, $K \in \mathcal{D}$.

$$X_{k+1} = \phi X_k + \Gamma U_k + W_k$$

$$Y_k = C X_k + V_k$$

$$U_k = - K Y_k$$

$$E(W_k \; W_i^T) = W\delta_{ki} \qquad E(V_k \; V_i^T) = V\delta_{ki} \qquad E(X_o \; X_o^T) = S_o$$

$$E(W_k) = 0 \qquad E(V_k) = 0 \qquad E(W_k \; V_j^T) = E(W_k \; X_o^T) = E(V_k \; X_o^T) = 0$$

$$J_N(K) = \frac{1}{2(N+1)} \sum_{K=0}^{N} E(X_{k+1}^T \, Q \, X_{k+1} + U_k^T \, R \, U_k)$$

$$J(K) = \lim J_N(K) < \infty \qquad K \in \mathcal{D}$$

EXAMPLE

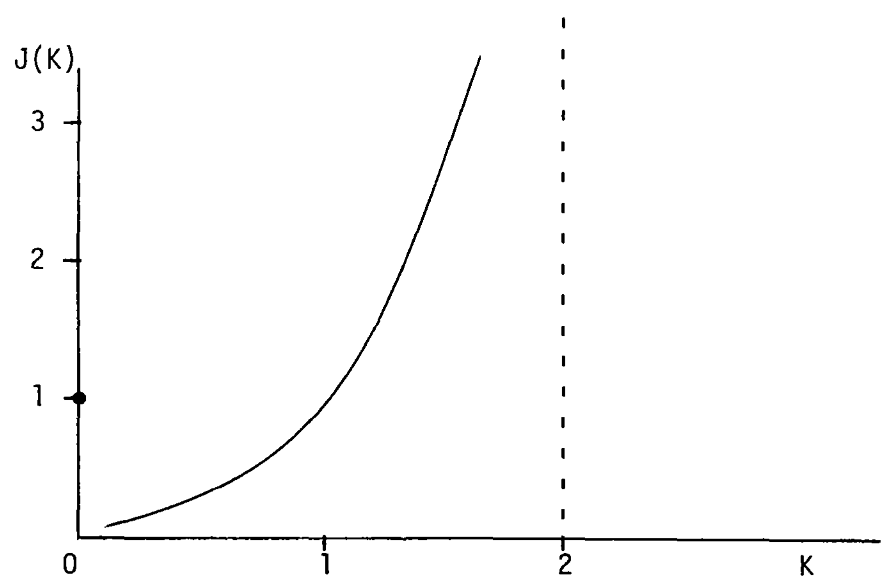
Some important characteristics of the stochastic optimization problem posed are illustrated in a simple first-order example. In this example, the domain of optimization $\mathcal{D}$ is the semi-open interval (0, 2), while the set of stabilizing gains S consists of the open interval (0, 2). The system is completely controllable and output stabilizable. However, as illustrated by the example, output stabilizability alone does not guarantee the existence of a solution to the optimization problem. The cost function J(K), for this example, has no minimum in $\mathcal{D}$ or in S. Furthermore, the example illustrates that the continuity of the cost function J(K) over its domain $\mathcal{D}$ is not guaranteed, as K = 0 is a point of discontinuity. Therefore, it is desirable to determine conditions under which an optimal solution exists.

$$X_{k+1} = X_k + U_k \qquad Y_k = X_k + V_k$$

$$Q = 1 \qquad R = 0 \qquad V = 1 \qquad W = 0 \qquad S_0 = 1$$

$$\Gamma^T Q \Gamma + R = 1 > 0 \qquad C W C^T + V = 1 > 0$$

$$J(K) = \begin{cases} \dfrac{K}{2-K} & 0 < K < 2 \\ 1 & K = 0 \end{cases}$$



OUTPUT FEEDBACK EXISTENCE CONDITIONS

As illustrated by the example, the domain of optimization is not necessarily a closed set, and can be unbounded, although S is always open. Thus, it is necessary to determine conditions under which the minimum cost is attained at an interior point of S. Such conditions which guarantee the existence of a solution to the optimization problem are shown below. Under these conditions, the domain of optimization $\mathcal{D}$ coincides with the stability set S, which insures that the optimal gain stabilizes the closed-loop system. On the other hand, it can be shown that the cost function J(K) is always continuous on S [10]. Note that the example considered previously fails to satisfy the condition $W \geq \varepsilon \, \Gamma \, \Gamma^T$, but satisfies all the remaining conditions. While the conditions 1, 2, and 3 ensure the existence of a stable global minimum, they are not necessary for the existence of a solution to the optimization problem. However, the class of optimization problems covered is quite broad, and because the existence conditions are expressed in terms of known system parameter matrices, verification is a simple task. Note that the measurement noise and control penalty terms are not necessary for existence, which is a major difference between the discrete and continuous output feedback problems. Also note that Q and W need not be positive definite, but must satisfy 1. Condition 1 is intriguing, as it corresponds to a method of improving robustness in control and filter designs [11]. The uniqueness of the solution is not ensured, except for special cases such as **full-state** feedback.

SUFFICIENT CONDITIONS FOR EXISTENCE:

1. FOR SOME $\varepsilon > 0$ $\quad Q \geq \varepsilon \, C^T C \quad W \geq \varepsilon \, \Gamma \, \Gamma^T$

2. $\Gamma^T Q \, \Gamma + R > 0 \quad C \, W \, C^T + V > 0$

3. (C, ϕ, Γ) IS OUTPUT STABILIZABLE

LET 1 AND 2 HOLD. J(K) HAS A STABLE MINIMUM IF, AND ONLY IF, (C, ϕ, Γ) IS OUTPUT STABILIZABLE

For gains which stabilize the closed-loop system, the cost function $J(K)$ can be expressed more explicitly in terms of K, as shown below. An expression which provides more insight can be obtained by considering the incremental cost $\Delta J(K, \Delta K)$. As the incremental cost is the total change in the cost due to a change ΔK in the gain, the optimization problem can also be treated as that of finding a ΔK^* which minimizes the incremental cost for a fixed $K \in \mathcal{S}$. Due to the almost quadratic form of the incremental cost, a "natural" direction is the one which would minimize the incremental cost if it were actually quadratic in ΔK. The following theorem exploits this direction, $d(K)$.

Theorem: Let the existence conditions 1, 2, and 3 hold, and K_0 be in $\mathcal{S}$. Then there exist $\beta > 0$, a sequence $\{K_i,\ i \geq 0\}$, and a limit point, say K^*, of the sequence such that

$$J(K_i) \downarrow J(K^*) \quad \text{and} \quad \frac{\partial J}{\partial K}(K_i) \rightarrow \frac{\partial J}{\partial K}(K^*) = 0$$

$$K_{i+1} = K_i + \alpha\, d(K_i)$$

$$d(K) = \hat{P}(K)^{-1}\, \Gamma^T\, P(K)\, \phi\, S(K)\, C^T\, \hat{S}(K)^{-1} - K$$

whenever $0 < \alpha \leq \beta$.

$$J(K) = \frac{1}{2}\,\mathrm{tr}\left\{P(K)\, W\right\} + \frac{1}{2}\,\mathrm{tr}\left\{K^T\, \hat{P}(K)\, K\, V\right\} \qquad K \in \mathcal{S}$$

$$P(K) = \phi(K)^T\, P(K)\, \phi(K) + C^T\, K^T\, R\, K\, C + Q$$

$$S(K) = \phi(K)\, S(K)\, \phi(K)^T + \Gamma\, K\, V\, K^T\, \Gamma^T + W$$

$$\hat{P}(K) = \Gamma^T\, P(K)\, \Gamma + R \qquad \hat{S}(K) = C\, S(K)\, C^T + V$$

$$\begin{aligned}\Delta J(K,\Delta K) &= J(K+\Delta K) - J(K) \\ &= \frac{1}{2}\,\mathrm{tr}\Big\{2\, \Delta K^T \left[\hat{P}(K+\Delta K)\, K\, \hat{S}(K) - \Gamma^T\, P(K+\Delta K)\, \phi\, S(K)\, C^T\right] \\ &\quad + \Delta K^T\, \hat{P}(K+\Delta K)\, \Delta K\, \hat{S}(K)\Big\} \qquad K,\ K+\Delta K \in \mathcal{S}\end{aligned}$$

NECESSARY CONDITIONS

$$\hat{P}(K^*)\, K^*\, \hat{S}(K^*) = \Gamma^T\, P(K^*)\, \phi\, S(K^*)\, C^T \qquad K^* \in \mathcal{S}$$

OUTPUT FEEDBACK ALGORITHM

Convergence **Theorem:** Let $\{K_i,\ i \geq 0\}$ be a sequence of gains obtained from the algorithm, starting with $K_0 \in \mathcal{S}$. Then, any limit point, say K^*, satisfies the necessary conditions for optimality, stabilizes the closed-loop system, and $J(K_i) \downarrow J(K^*)$.

1. CHOOSE $K_0 \in \mathcal{S} \quad \alpha_0 = 1 \quad z > 1 \quad i = 0$

2. SOLVE THE LYAPUNOV EQUATIONS

   $$P(K_i) = \phi(K_i)^T\, P(K_i)\, \phi(K_i) + C^T K_i^T R K_i C + Q$$

   $$S(K_i) = \phi(K_i)\, S(K_i)\, \phi(K_i)^T + \Gamma K_i V K_i^T \Gamma^T + W$$

   IF $P(K_i)$ OR $S(K_i)$ IS NOT NON-NEGATIVE DEFINITE GO TO 5

3. COMPUTE $d(K_i)$, K_{i+1}

   $$d(K_i) = \hat{P}(K_i)^{-1}\, \Gamma^T\, P(K_i)\, \phi\, S(K_i)\, C^T\, \hat{S}(K_i)^{-1} - K_i$$

   $$K_{i+1} = K_i + \alpha_i\, d(K_i)$$

4. COMPUTE THE COST $J(K_i)$

   $$J(K_i) = \frac{1}{2}\, \mathrm{tr}\left\{P(K_i)\, W\right\} + \frac{1}{2}\, \mathrm{tr}\left\{K_i^T\, P(K_i)\, K_i\, V\right\}$$

   IF $i = 0$, SET $i = 1$ AND GO TO 2.

   IF $J(K_i)-J(K_{i-1}) < -\frac{1}{4}\, \alpha_{i-1}(2-\alpha_{i-1})\ \mathrm{tr}\left\{d(K_{i-1})^T \hat{P}(K_{i-1}) d(K_{i-1}) \hat{S}(K_{i-1})\right\}$ GO TO 6

5. REDUCE α

   $\alpha_i = \alpha_i / z \quad K_i = K_{i-1} \quad d(K_i) = d(K_{i-1})$

   $K_{i+1} = K_i + \alpha_i\, d(K_i) \quad \alpha_{i+1} = \alpha_i \quad i = i+1$ GO TO 2

6. COMPUTE GRADIENT

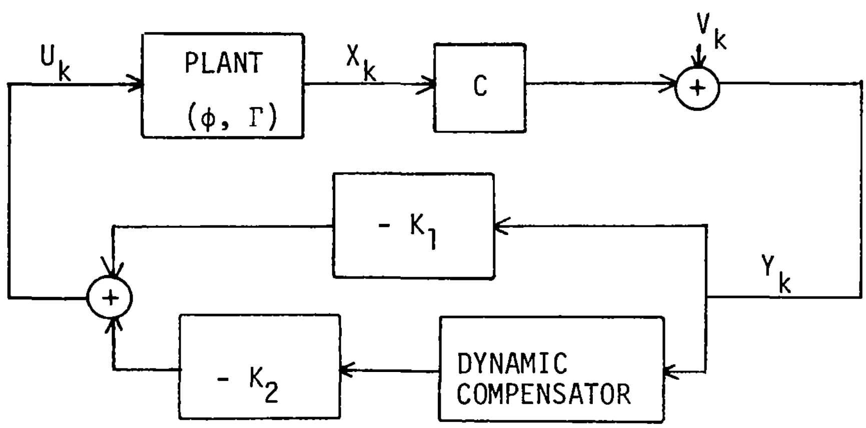
   $$\frac{\partial J}{\partial K}(K_i) = -\,\hat{P}(K_i)\, d(K_i)\, \hat{S}(K_i)$$

   IF $\left\| \frac{\partial J}{\partial K}(K_i) \right\| > \varepsilon_1$ OR $|J(K_i) - J(K_{i-1})| > \varepsilon_2$, $\alpha_{i+1} = \alpha_i$, $i = i+1$, GO TO 2

7. STOP

OPTIMAL DYNAMIC COMPENSATION

Most control systems for complex plants use some form of dynamic compensation. The dynamic compensator may simply consist of an integral feedback or a rate command structure, or may be a Kalman filter or an observer. Classical control designs make considerable use of dynamic compensation in the form of various filters, washout loops, etc. The basic form of a digital control system making use of dynamic compensation is shown below. The design of dynamic compensation in an optimal control setting can be imbedded into the optimal output feedback formulation by augmenting the state with the compensator states [12,13]. In this form, the order of the dynamic compensator is a design parameter and can be selected so as to obtain a low order, easily implemented compensator. For systems which are not stabilizable with the available measurements, such as some cases of flutter suppression, dynamic compensation is a necessary rather than a simply desirable structure [14]. The design of the dynamic compensator can be obtained using the output feedback algorithm presented earlier.



$$X_{k+1} = \phi X_k + \Gamma U_k + W_k \qquad Y_k = C X_k + V_k$$

$$Z_{k+1} = \phi_z Z_k + \Gamma_z Y_k \qquad U_k = - K_1 Y_k - K_2 Z_k$$

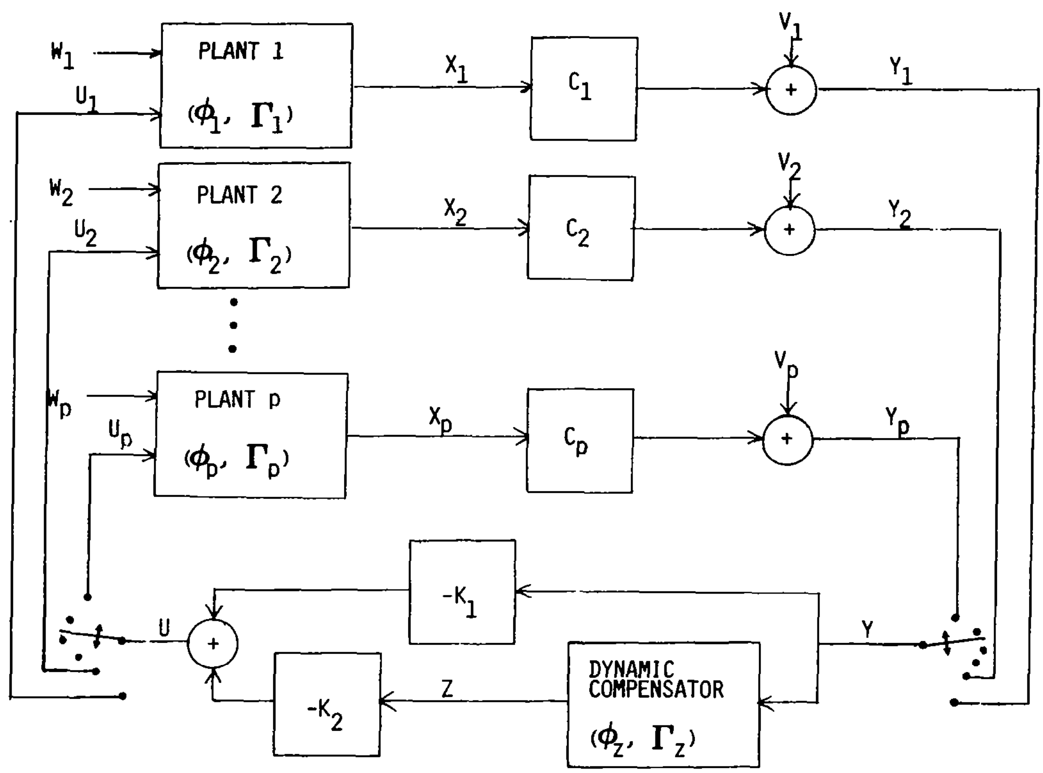
$$\begin{pmatrix} X_{k+1} \\ Z_{k+1} \end{pmatrix} = \begin{pmatrix} \phi & 0 \\ 0 & \phi_{zo} \end{pmatrix} \begin{pmatrix} X_k \\ Z_k \end{pmatrix} + \begin{pmatrix} \Gamma & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} U_k \\ \mu_k \end{pmatrix} + \begin{pmatrix} W_k \\ W_{zk} \end{pmatrix}$$

$$\bar{Y}_k = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X_k \\ Z_k \end{pmatrix} + \begin{pmatrix} V_k \\ 0 \end{pmatrix} \qquad \begin{pmatrix} U_k \\ \mu_k \end{pmatrix} = - \begin{pmatrix} K_1 & K_2 \\ -\Gamma_z & -(\phi_z - \phi_{zo}) \end{pmatrix} \bar{Y}_k$$

ADVANTAGES OF THE MULTIPLE-MODEL OUTPUT FEEDBACK APPROACH

While output feedback introduces significant flexibility in the structure of a control law, it does not directly address some of the objectives and requirements encountered in designing control laws. The multiple-model output feedback approach provides a design method which can be used to obtain important design requirements while preserving all the advantages inherent in output feedback. Some of the advantages of the approach follow.

- PRESERVES ALL THE ADVANTAGES OF OUTPUT FEEDBACK
- PROVIDES A DESIGN METHOD FOR ROBUST CONTROL LAWS
- PROVIDES A DESIGN METHOD FOR MULTIPLE CRITERIA
- PROVIDES A DESIGN METHOD FOR ACTUATOR FAILURE ACCOMMODATION
- PROVIDES A DESIGN METHOD FOR SENSOR FAILURE ACCOMMODATION



FORMULATION OF THE MULTIPLE-MODEL OUTPUT FEEDBACK PROBLEM

The multiple-model output feedback formulation considers the problem of designing a fixed control law to meet design objectives expressed in terms of various plant models, measurement models, and performance criteria as shown below. The control law structure can contain dynamic compensation and output feedback. For example, the design objective of insensitivity to variations in some plant parameters can be addressed by selecting plant models (ϕ_j, Γ_j) which include these variations. Some types of actuator, sensor, or other plant subsystem failures can be addressed in the design by appropriate selection of the parameters Γ_j, C_j, ϕ_j, V_j, and W_j. Various other design objectives can be addressed in a similar manner.

Let $\mathcal{S}_j$ be the set of gains which stabilizes the j^{th} plant model, while $\mathcal{D}_j$ is the set of gains for which the j^{th} cost remains finite. The intersection $\mathcal{S}$ of the $\mathcal{S}_j$'s determines the control gains which stabilize all the plant models, while the intersection $\mathcal{D}$ of the $\mathcal{D}_j$'s is the set on which the total cost $J(K)$ is finite. The optimization problem can be posed as: Find a gain K^* which stabilizes all the models (i.e., $K^* \epsilon \mathcal{S}$) and minimizes the cost $J(K)$, i.e., $J(K^*) \le J(K)$, $K \epsilon \mathcal{D}$.

$$X_{jk+1} = \phi_j X_{jk} + \Gamma_j U_{jk} + W_{jk} \qquad 1 \le j \le p$$

$$Y_{jk} = C_j X_{jk} + V_{jk} \qquad 1 \le j \le p$$

$$U_{jk} = - K Y_{jk} = - K C_j X_{jk} - K V_{jk}$$

$$J_j(K) = \lim_{N \to \infty} \frac{1}{2(N+1)} \sum_{k=0}^{N} E(X^T_{jk+1} Q_j X_{jk+1} + U^T_{jk} R_j U_{jk}) < \infty \qquad K \epsilon \mathcal{D}_j$$

$$J(K) = \sum_{j=1}^{p} \gamma_j J_j(K) \qquad \gamma_j > 0 \qquad K \epsilon \mathcal{D}$$

$$\mathcal{D} = \bigcap_{j=1}^{p} \mathcal{D}_j \qquad \mathcal{S} = \bigcap_{j=1}^{p} \mathcal{S}_j$$

MULTIPLE-MODEL OUTPUT FEEDBACK EXISTENCE CONDITIONS

As for the case of the (single-model) output feedback problem, the existence of a solution to the optimal control problem posed is not always ensured. However, as seen by the sufficient conditions given below, a solution does exist for a large class of optimization problems. It should be noted that the constraint that the optimal gain **stabilizes the closed-loop models excludes** problems where no stabilizing gain exists, so that this class of problems must be treated separately. The sufficient conditions for the problem posed can be obtained by extending the results for output feedback. It can be shown that the cost function $J(K)$ is always continuous on $\mathcal{S}$, but not necessarily on $\mathcal{D}$. However, for the class of problems satisfying the sufficient conditions, $\mathcal{D}$ and $\mathcal{S}$ are equal. The conditions given here are not necessary for the existence of a stable global minimum, and the uniqueness of a solution is not ensured. Nevertheless, the class of problems included is broad enough to cover most parameter sensitivity objectives, control, or sensor failure accommodation objectives, as well as other significant objectives.

SUFFICIENT CONDITIONS FOR EXISTENCE

1. FOR SOME $\varepsilon > 0$, AND ALL $j \le p$ $\quad Q_j \ge \varepsilon\, C_j^T C_j \quad W_j \ge \varepsilon\, \Gamma_j \Gamma_j^T$

2. $\Gamma_j^T Q_j \Gamma_j + R_j > 0 \qquad C_j W_j C_j^T + V_j > 0 \qquad 1 \le j \le p$

3. $\mathcal{S}$ IS NON-NULL

LET 1 AND 2 HOLD. J(K) HAS A STABLE MINIMUM IF, AND ONLY IF,

$\mathcal{S} = \bigcap_{j=1}^{p} \mathcal{S}_j$ CONTAINS AN ELEMENT

MULTIPLE-MODEL INCREMENTAL COST AND NECESSARY CONDITIONS

For gains which stabilize all the plant models, the cost function $J(K)$ can be expressed in terms of the gain K, as shown. Similarly, the incremental cost $\Delta J(K,\Delta K)$ or the change in the cost due to a change in the gain, is seen to resemble a quadratic form in ΔK. The necessary conditions can be easily obtained from the incremental cost by letting ΔK approach zero. The direction $d(K)$, which would minimize the incremental cost if it were a true quadratic form, is selected to obtain an algorithm. It is seen that the direction $d(K)$ is the solution of a linear equation which requires a larger number of computations than the output feedback case. Also note that setting p equal to one results in the output feedback equations.

$$J(K) = \frac{1}{2} \sum_{j=1}^{p} \gamma_j \Big[\mathrm{tr}\{P_j(K)\ W_j\} + \mathrm{tr}\{K^T\ \hat{P}_j(K)\ K\ V_j\}\Big] \qquad K \in S$$

$$P_j(K) = \phi_j(K)^T\ P_j(K)\ \phi_j(K) + C_j^T\ K^T\ R_j\ K\ C_j + Q_j$$

$$S_j(K) = \phi_j(K)\ S_j(K)\ \phi_J(K)^T + \Gamma_j\ K\ V_j\ K^T\ \Gamma_j^T + W_j$$

$$\hat{P}_j(K) = \Gamma_j^T\ P_j(K)\ \Gamma_j + R_j \qquad \hat{S}_j(K) = C_j\ S_j(K)\ C_j^T + V_j$$

$$\Delta J(K,\Delta K) = \frac{1}{2}\ \mathrm{tr}\Big\{2\ \Delta K^T \sum_{j=1}^{p} \gamma_j \Big[\hat{P}_j(K+\Delta K)\ K\ \hat{S}_j(K) - \Gamma_j^T\ P_j(K+\Delta K)\ \phi_j\ S_j(K)\ C_j^T\Big]$$

$$+ \sum_{j=1}^{p} \gamma_j\ \Delta K^T\ \hat{P}_j(K+\Delta K)\ \Delta K\ \hat{S}_j(K)\Big\} \qquad K \quad K+\Delta K \in S$$

$$\frac{\partial J}{\partial K}(K) = \sum_{j=1}^{p} \gamma_j\ \hat{P}_j(K)\ K\ \hat{S}_j(K) - \Gamma_j^T\ P_j(K)\ \phi_j\ S_j(K)\ C_j^T$$

$$\sum_{j=1}^{p} \gamma_j\ \ \hat{P}_j(K)\ d(K)\ \hat{S}_j(K) = -\frac{\partial J}{\partial K}(K)$$

NECESSARY CONDITIONS

$$\sum_{j=1}^{p} \gamma_j\ \hat{P}_j(K^*)\ K^*\ \hat{S}_j(K^*) = \sum_{j=1}^{p} \gamma_j\ \Gamma_j^T\ P_j(K^*)\ \phi_j\ S_j(K^*)\ C_j^T \qquad K^* \in S$$

MULTIPLE-MODEL OUTPUT FEEDBACK ALGORITHM

1. CHOOSE $K_0 \in S$, $\alpha_0 = 1$, $z > 1$, $i = 0$

2. SOLVE THE LYAPUNOV EQUATIONS FOR $j = 1, \ldots, p$

$$P_j(K_i) = \phi_j(K_i)^T P_j(K_i) \phi_j(K_i) + C_j^T K_k^T R_j K_i C_j + Q_j$$

$$S_j(K_i) = \phi_j(K_i) S_j(K_i) \phi_j(K_i)^T + \Gamma_j K_i V_j K_i^T \Gamma_j^T + W_j$$

IF $P_j(K_i)$ OR $S_j(K_i)$ IS NOT NON-NEGATIVE DEFINITE GO TO 5

3. SOLVE FOR $d(K_i)$, K_{i+1}

$$\sum_{j=1}^{p} \gamma_j \hat{P}_j(K_i) d(K_i) \hat{S}_j(K_i) = -\frac{\partial J}{\partial K}(K_i)$$

$$K_{i+1} = K_i + \alpha_i d(K_i)$$

4. COMPUTE THE COST $J(K_i)$

$$J(K_i) = \frac{1}{2} \sum_{j=1}^{p} \gamma_i \left[tr\{P_j(K_i) W_j + tr\{K_i^T P_j(K_i) K_i V_j\}\right]$$

IF $i = 0$, SET $i = 1$ AND GO TO 2

IF $J(K_i)-J(K_{i-1}) < -\frac{1}{4} \alpha_{i-1}(2-\alpha_{i-1}) \sum_{j=1}^{p} \gamma_j \, tr\left\{d(K_{i-1})^T \hat{P}_j(K_{i-1}) d(K_{i-1}) \hat{S}_j(Ki-1)\right\}$

GO TO 6

5. REDUCE α

$\alpha_i = \alpha_i/z$ $\quad K_i = K_{i-1}$ $\quad d(K_i) = d(K_{i-1})$

$K_{i+1} = K_i + \alpha_i d(K_i)$ $\quad \alpha_{i+1} = \alpha_i$ $\quad i = i+1$ GO TO 2

6. CHECK CONVERGENCE

IF $\left\| \frac{\partial J}{\partial K}(K_i) \right\| > \varepsilon_1$ OR $|J(K_i) - J(K_{i-1})| > \varepsilon_2$, $\alpha_{i+1} = \alpha_i$, $i = i+1$, GO TO 2

7. STOP

DECENTRALIZED CONTROL FORMULATION

It is well known that the optimal decentralized control problem for linear plants with Gaussian statistics and quadratic cost criteria does not necessarily result in a linear system, and when constrained to linear systems may result in infinite order systems [15-17]. Given these negative results, it is natural to constrain the class of decentralized controllers to linear systems of fixed finite order, i.e., the class of decentralized controllers with fixed-order dynamic compensators as local controllers [18]. However, since the dynamic compensator problem can be imbedded into the output feedback problem, it suffices to consider the class of decentralized output feedback controllers. Furthermore, the decentralized output feedback problem can be posed as a constrained output feedback problem, where the gain K is restricted to block diagonal form [18]. Let $\mathcal{S}$ be the collection of block diagram gains (of appropriate dimensions) which stabilize the decentralized system, while $\mathcal{D}$ is the set of block diagonal gains for which the cost $J(K)$ is finite. Then the problem can be posed as: Find a stabilizing gain $K^* \in \mathcal{S}$ which minimizes the cost function over $\mathcal{D}$, i.e., $J(K^*) \leq J(K)$, $K \in \mathcal{D}$. It can be shown that if a block diagonal stabilizing gain exists with $\Gamma^T Q\Gamma + R$ and $CWC^T + V$ positive definite, and if there exists some $\varepsilon > 0$ such that $Q > \varepsilon C^TC$ and $W > \varepsilon \Gamma \Gamma^T$, then the optimal decentralized control problem posed has a solution. The necessary conditions are easily obtained from the incremental cost [18].

$$X_{k+1} = \phi X_k + \sum_{\ell=1}^{L} \Gamma_\ell U_{\ell k} + W_k$$

$$Y_{\ell k} = C_\ell X_k + V_{\ell k} \qquad 1 \leq \ell \leq L$$

$$U_{\ell k} = - K_\ell U_{\ell k} \qquad 1 \leq \ell \leq L$$

$$J(K) = \lim_{N\to\infty} \frac{1}{2(N+1)} \sum_{k=0}^{N} \sum_{\ell=1}^{L} E(X_{k+1}^T Q_\ell X_{k+1} + U_{\ell k}^T R_\ell U_{\ell k})$$

$$X_{k+1} = \phi X_k + \Gamma U_k + W_k \qquad Y_k = C X_k + V_k$$

$$U_k = - K Y_k$$

$$C = \begin{pmatrix} C_1 \\ \vdots \\ C_L \end{pmatrix} \qquad \Gamma = (\Gamma_1, \ldots, \Gamma_L)$$

$$K = \text{BLOCK DIAG} \left\{ K_\ell \qquad 1 \leq \ell \leq L \right\}$$

$$J(K) = \lim_{N\to\infty} \frac{1}{2(N+1)} \sum_{k=0}^{N} E(X_{k+1}^T Q X_{k+1} + U_k^T R U_k) < \infty \qquad K \in \mathcal{D}$$

MULTIPLE-MODEL DECENTRALIZED CONTROL ALGORITHM

One approach to obtain an algorithm for the decentralized control problem posed is to close the control loop over all the local controllers, except one, solve the resulting unconstrained output feedback problem, then iterate on the next local controller. While this method does not make use of some of the analytical tools developed, which would result in a more efficient algorithm, a solution to the combined decentralized **multiple-model** output feedback can be obtained using the previously developed algorithm for **multiple-model** problems.

1. CHOOSE A BLOCK DIAGONAL K^o in S $\quad i = 1 \quad \ell = 1$

2. SOLVE THE **MULTIPLE-MODEL** OUTPUT FEEDBACK PROBLEM FOR THE SYSTEMS

$$\left\{(C_{\ell j},\ \phi_j - \sum_{\ell' \neq \ell} \Gamma_{\ell' j} K^i_{\ell'} C_{\ell' j},\ \Gamma_{\ell j}) \quad j = 1, 2, \ldots, p\right\}$$

WITH WEIGHTING MATRICES $\left\{Q_j + \sum_{\ell' \neq \ell} C^T_{\ell' j} K^{i^T}_{\ell'} R_{\ell' j} K^i_{\ell'} C_{\ell'} \quad j = 1, \ldots, p\right\}$

$\left\{R_{\ell j},\ \ j = 1, \ldots, p\right\}$ AND STATISTICS $\left\{V_{\ell j} \quad j = 1, 2, \ldots, p\right\}$

$\left\{W_j - \sum_{\ell' \neq \ell} \Gamma^T_{\ell' j} K^{i^T}_{\ell'} V_{\ell' j} K^i_{\ell'} \Gamma_{\ell' j} \quad j = 1, \ldots, p\right\}$ TO OBTAIN K^{i+1}

3. IF $\ell = L$, GO TO 4

SET $K^i_\ell = K^{i+1}_\ell$ $\quad \ell = \ell + 1$ $\quad$ GO TO 2

4. SET $K^{i+1}_\ell = K^i_\ell$ $\quad 1 \le \ell \le L$

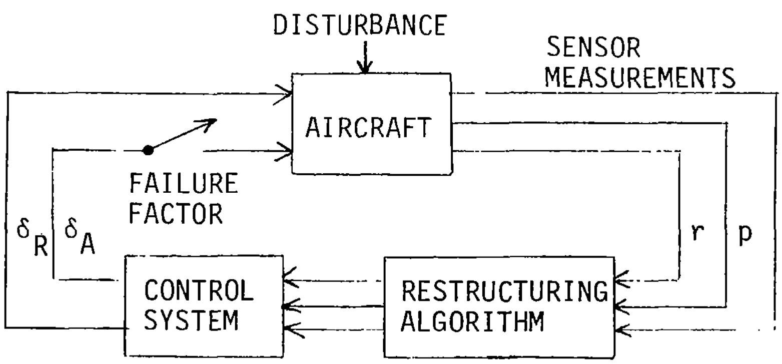
IF $|J(K^{i+1}) - J(K^i)| \le \varepsilon_1$ AND $\left\|\frac{\partial J}{\partial K}(K^{i+1})\right\| \le \varepsilon_2$ $\quad$ STOP

5. $i = i + 1 \quad \ell = 1$ $\quad$ GO TO 2

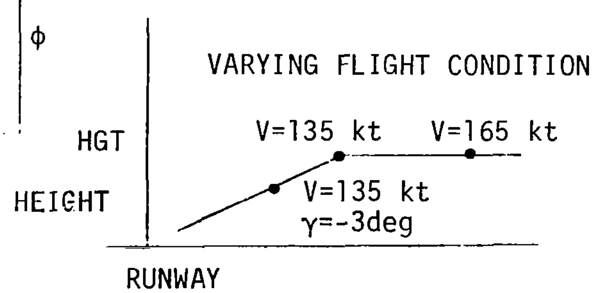
AN APPLICATION TO RESTRUCTURABLE CONTROLS

The optimal stochastic multiple-model, decentralized control, output feedback and dynamic compensation problems formulated provide powerful techniques to investigate a large class of control system design problems. The stochastic output feedback, dynamic compensation, and decentralized control problems provide a wealth of control structures; however, the "best" structure(s) for a given design problem are not determined and depend on the practical constraints. On the other hand, the multiple-model formulation provides a powerful technique to describe design objectives in an optimal control setting, with computable algorithms and implementable structures.

As an illustration, the combined multiple-model decentralized control algorithm is used to design a simple aircraft control law which accommodates some types of control actuator failures. While the performance of a control law under normal conditions is a primary design goal, a practical control design must also consider the implications of various scenarios, such as actuator failures. The restructurable control problem addresses methods of modifying the structure of the law to accommodate failures. However, the detection of the exact nature of the failure requires a period of time. Depending on the actual failure, during this period of time the aircraft may be forced into a condition from which recovery is difficult, and sometimes not possible. Therefore, it is reasonable to restructure the control system in stages. As soon as the existence of a failure is known, or even highly likely, the system may be restructured into a control law which can accommodate a large number of failed components. This first stage restructuring can provide the valuable time necessary to identify the exact failure, decide the best second-stage structure, and implement it before the aircraft is forced into a possibly irrecoverable condition. The multiple-model formulation provides a control design method where the law is at least stable in the failed condition as well as under normal circumstances, when possible. Since the control law is stable for the case of no failure, a false alarm does not produce harmful effects. In the following example, a wing leveler with dynamic compensation and a decentralized structure is modeled with normal and failed aileron for failure accommodation, and at two airspeeds to provide insensitivity.



FAILURE ACCOMMODATION



FLIGHT CONTROL DESIGN EXAMPLE

The advantages offered by the multiple-model decentralized approach are demonstrated using an aircraft digital flight control system design problem. The control problem is the simplified design of the lateral dynamics, inner-loop control system for the NASA ATOPS research aircraft (a Boeing 737). The aircraft model includes the body-axis states, v, p, r, and ϕ. Also included in the model are aileron and rudder actuators dynamics, a one-state dynamic compensator, and aileron and rudder control states caused by weighting the control difference in the quadratic cost function. Noisy sensor measurements are p, r, and ϕ. The dynamic compensator state and control states are noise-free measurements. The dynamic compensator state is quadratically weighted to "follow" the aircraft v state. The closed-loop eigenvalues using optimal output feedback at one trimmed flight condition (V_0 = 135 kn, Wt = 85,000 lbs, h_0 = 1,000 ft) are shown below. The other table shows the closed-loop eigenvalues for the multiple-model, decentralized design at the same flight condition with the same quadratic weights and noise covariances. The Dutch roll mode has the lowest damping in both designs.

OUTPUT FEEDBACK DESIGN

MAPPED EIGENVALUES (135 KN)	
REAL	IMAG.
- .521	0.00
- .903	1.47
- .903	-1.47
- 1.53	1.72
- 1.53	-1.72
- 1.25	0.92
- 1.25	-0.92
-14.4	0.00
-36.5	0.00

MULTIPLE-MODEL DECENTRALIZED DESIGN

MAPPED EIGENVALUES (135 KN)	
REAL	IMAG.
- .436	0.00
- .925	1.93
- .925	-1.93
- 1.08	1.22
- 1.08	-1.22
- 1.23	0.00
- 2.62	0.00
-13.4	0.00
-42.1	0.00

FEEDBACK GAINS FOR SINGLE- AND MULTIPLE-MODEL DESIGNS

The lateral dynamics flight control design for the multiple-model case uses four models: two models at 135 kn and 165 kn with the aileron operational, and two models at 135 kn and 165 kn with the aileron failed. The design is decentralized as four gains in the output feedback gain matrix are forced to be zero. The controls are the dynamic compensator control μ, aileron rate $\dot{\delta A}$, and rudder rate $\dot{\delta R}$. In the multiple-model decentralized design, no control states are fed back to the dynamic compensator, and aileron and rudder control state crossfeed gains are forced to be zero. The primary differences in the two gain matrices are the ϕ and p gains to $\dot{\delta R}$ which change sign. The fixed-gain multiple-model design stabilizes all four models. The single-model design causes the closed-loop system with the aileron failed to be unstable.

OUTPUT FEEDBACK DESIGN

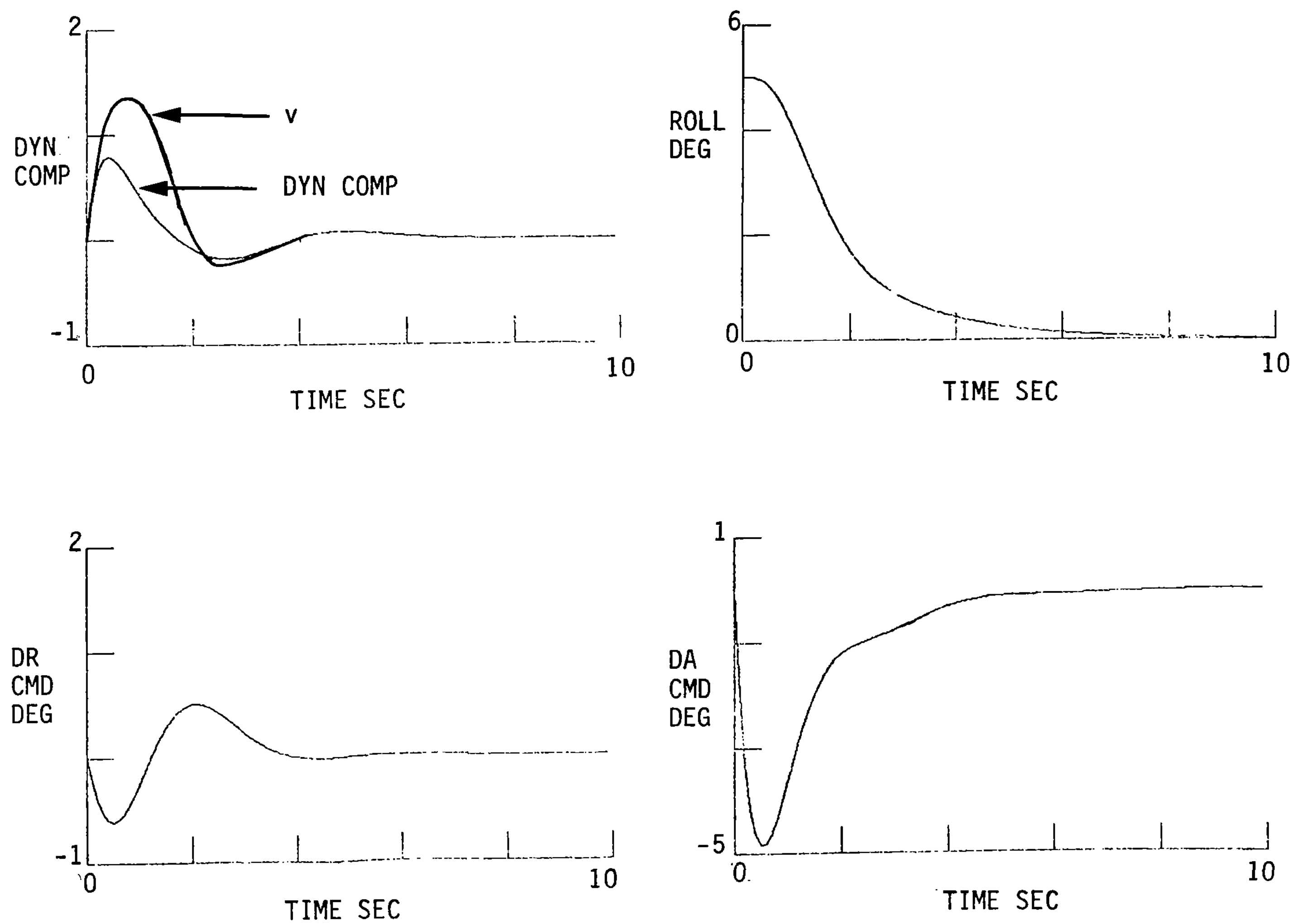
CONTROL GAIN MATRIX						
	COMP.	p	r	ϕ	δA	δR
$\delta\mu$	-0.769	0.0032	-3.47	0.829	0.661	0.721
$\dot{\delta A}$	1.92	-3.94	-7.09	-3.96	-2.94	0.698
$\dot{\delta R}$	-0.250	-0.499	3.77	-0.447	0.0055	-2.77

MULTIPLE-MODEL DECENTRALIZED DESIGN

CONTROL GAIN MATRIX						
	COMP.	p	r	ϕ	δA	δR
$\delta\mu$	-0.80	-0.527	-4.58	0.731	0.	0.
$\dot{\delta A}$	0.104	-2.49	-4.51	-1.96	-1.94	0.
$\dot{\delta R}$	-0.516	0.721	7.74	0.860	0.	-4.05

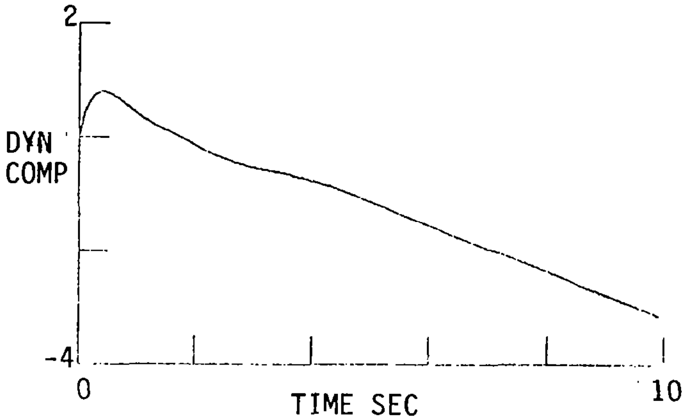
SINGLE-MODEL DESIGN SIMULATION - NO FAILURES

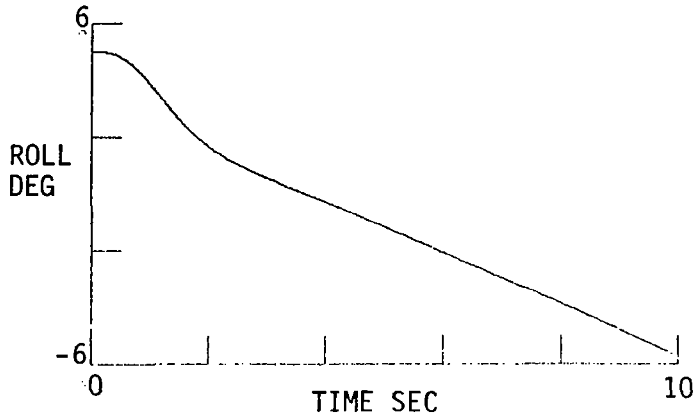
A linear simulation of the **single-model** optimal output feedback design is shown below. Roll attitude is initially 5 deg at the beginning of the simulation and is smoothly returned to zero by the control system. Lateral velocity v is kept **small, and the dynamic compensator state is similar to v.**

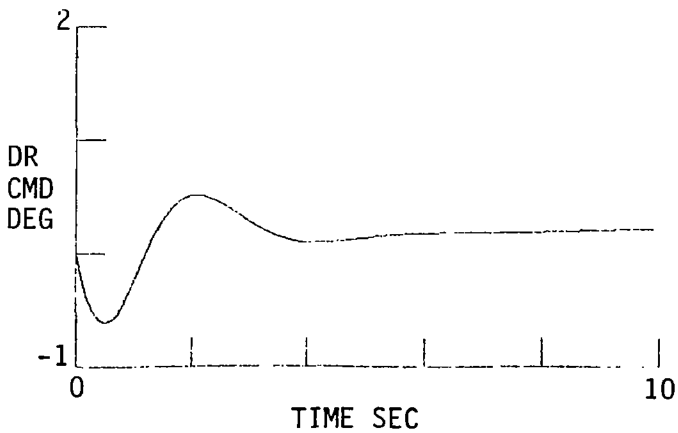


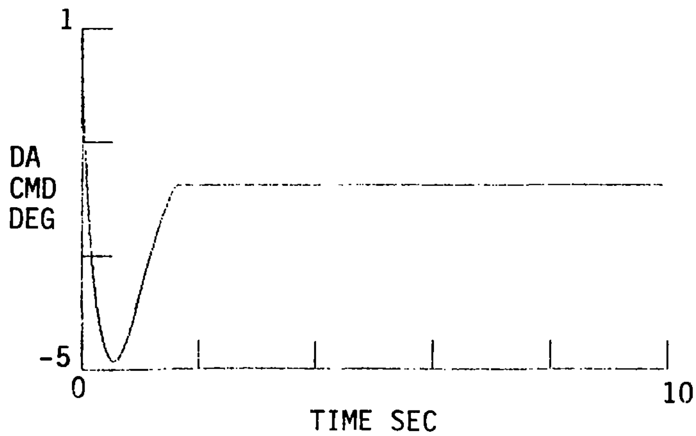
SINGLE-MODEL DESIGN SIMULATION - FAILED AILERON

The linear simulation shows the aileron failing 1.5 sec into the simulation and remaining fixed at approximately 2 deg. The closed-loop aircraft system is unstable and roll attitude is seen to diverge.



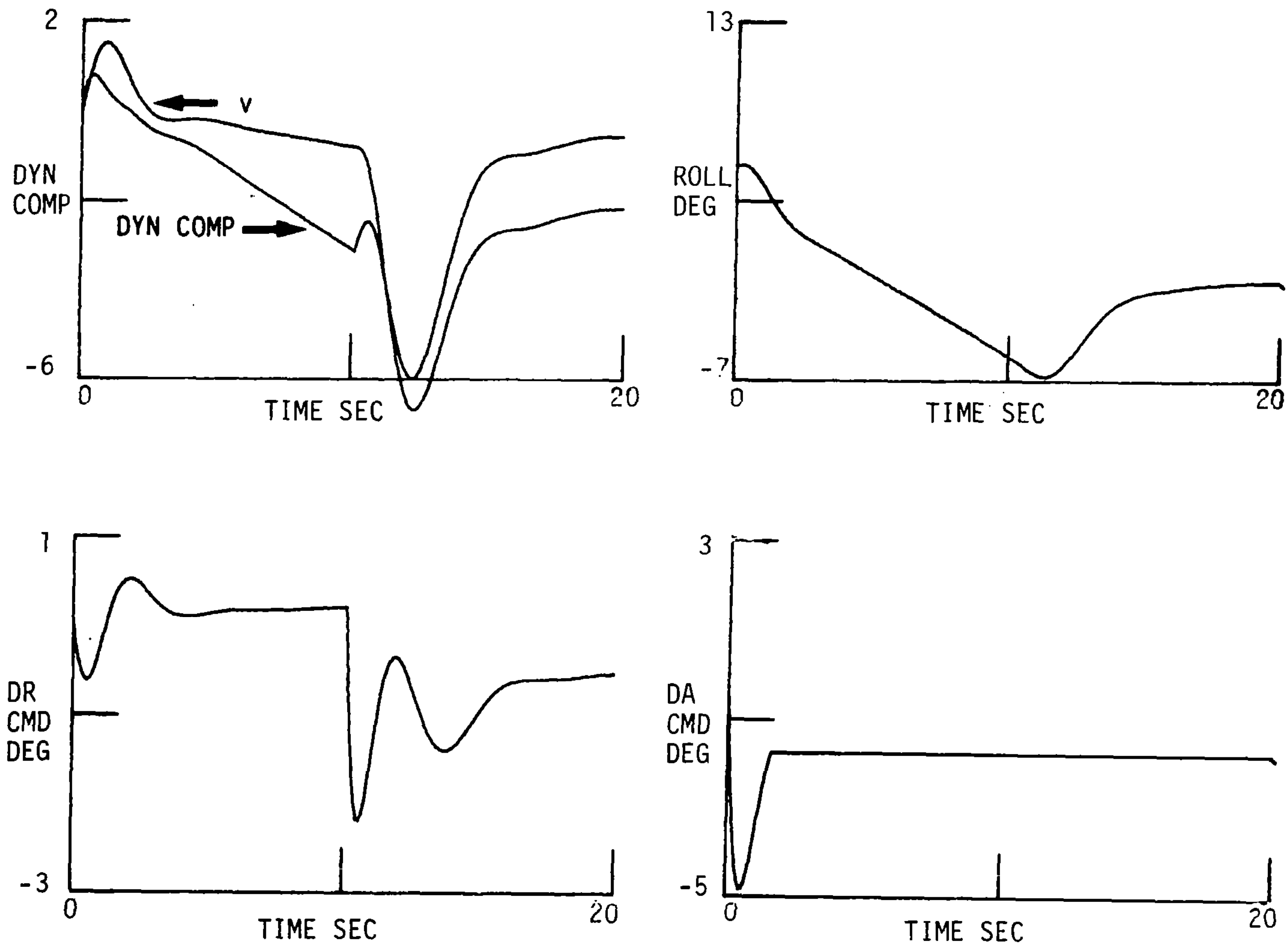






FIRST-STAGE RESTRUCTURING: MULTIPLE-MODEL DECENTRALIZED DESIGN - FAILED AILERON

The linear simulation shows the aileron failing at 1.5 sec into the simulation and remaining fixed at that level in the following period. The **single-model** output feedback design controls the aircraft until 10 sec. As the closed-loop system is unstable in this condition, the aircraft continues to roll past the level wings condition. It is assumed that by 10 sec., or 8.5 sec. after the aileron failure, the decision that a failure has occurred is made, and the first stage restructuring is engaged. The control law simulated after 10 sec. is the **multiple-model** decentralized design. As shown by the simulation, the restructured control arrests the roll of the aircraft, and brings it to a **non-zero but** easily manageable and stable bank angle, providing the time necessary for the **second-stage** restructuring.



REFERENCES

1. Axaster, S., "Sub-Optimal Time Variable Feedback Control of Linear Dynamic Systems with Random Inputs", Inter. J. of Control, Vol. 4, No. 6, Dec. 1966, pp. 549-566.

2. Rekasius, Z. V., "Optimal Linear Regulators with Incomplete State Feedback", IEEE Trans. Automatic Control, Vol. AC-12, June 1967, pp. 296-299.

3. Kosut, R. L., "Suboptimal Control of Linear Time-Invariant Systems Subject to Control Structure Constraints", IEEE Trans. Automatic Control, Vol. AC-15, October 1970, pp. 557-563.

4. Levine, W. S., and Athans, M., "On the Determination of the Optimal Constant Output Feedback Gains for Linear Multivariable Systems", IEEE Trans. Automatic Control, Vol. AC-15, February 1970, pp. 44-48.

5. Kurtaran, B., and Sidar, M., "Optimal Instantaneous Output-Feedback Controllers for Linear Stochastic Systems", Inter. J. of Control, Vol. 19, No. 4, April 1974, pp. 154-157.

6. O'Reilly, J., "Optimal Instantaneous Output-Feedback Controllers for Discrete-Time Linear Systems with Inaccessible State", Inter. J. of System Science, Vol. 9, No. 1, Jan. 1978, pp. 9-16.

7. Sobel, K., and Kaufman, H., "Application of Stochastic Optimal Reduced State Feedback Gain Computation Procedures to the Design of Aircraft Gust Alleviation Controllers", Proc. of IFAC Conference, vol. 2, Helsinki, Finland, 1978, pp. 1127-1233.

8. Horisberger, H. P., and Belanger, P. R., "Solution of the Optimal Constant Output Feedback Problem by Conjugate Gradients", IEEE Trans. Automatic Control, Vol. AC-19, No. 4, August 1974, pp. 434-435.

9. Athans, M., "The Role and Use of the Stochastic Linear-Quadratic-Gaussian Problem in Control System Design", IEEE Trans. Automatic Control, Vol. AC-16, December 1971, pp. 529-552.

10. Halyo, N., and Broussard, J. R., "A Convergent Algorithm for the Stochastic Infinite-Time Discrete Optimal Output Feedback Problem", Proc. of the 1981 JACC, Charlottesville, VA, June 1981, p. WA-1E.

11. Doyle, J. C., and Stein, G., "Robustness with Observers", IEEE Trans. Automatic Control, Vol. AC-24, August 1979, pp. 607-611.

12. Johnson, T. L., and Athans, M., "On the Design of Optimal Constrained Dynamic Compensators for Linear Constant Systems", IEEE Trans. Automatic Control, Vol. AC-15, December 1970, pp. 658-660.

13. Mendel, J. M., and Feather, J., "On the Design of Optimal Time-Invariant Compensators for Linear Stochastic Time-Invariant Systems", IEEE Trans. Automatic Control, Vol. 20, No. 5, October 1975, pp. 653-656.

14. Broussard, J. R., and Halyo, N., "Active Flutter Suppression Using Optimal Output Feedback Digital Controllers", NASA CR-165939, May 1982.

15. Witsenhausen, H. S., "A Counterexample in Stochastic Optimum Control", SIAM J. of Control, Vol. 6, No. 1, Feb. 1968, pp. 131-147.

16. Whittle, P., and Rudge, J. F., "The Optimal Linear Solution of a Symmetric Team Control Problem", J. of Applied Probability, Vol. 11, No. 2, June 1974, pp. 337-381.

17. Sandell, N. R., Varaiya, P., Athans, M., and Safonov, M. G., "Survey of Decentralized Control Methods for Large Scale Systems", IEEE Trans. on Automatic Control, Vol. AC-23, No. 2, April 1978, pp. 108-128.

18. Caglayan, A. K., Halyo, N., and Broussard, J. R., "The Use of the Optimal Output Feedback Algorithm in Integrated Control System Design", Proceedings of the IEEE 1983 National Aerospace and Electronics Conference NAECON 1983, Vol. 2, 1983, pp. 1242-1251.